Josephus problem as a multidisciplinary exercise source

Tomáš G. Roskovec[1], Petr Chládek[2], Jan Fiala[3]

Abstract: Josephus problem is an ancient mathematical problem based on the historical events lived and described by Flavius Josephus. As the problem description is rather vague, there is a traditional interpretation and some alternative versions. Our purpose is to show several different mathematical fields that can be applied to the solution with emphasis on usage in education and teaching, as the formulation of the problem is an ideal tool to introduce new methods and fields to students.

Keywords: Josephus problem, modular arithmetic, star polygons, algorithms
JEL Classification: C60, C88

1 Introduction

The main focus of our paper is to demonstrate the Josephus problem as an exciting topic for exercises solved by directly using different mathematical fields. We emphasize these points as exercises.

1.1 History introduction

Josephus Flavius (c. 37 to c. 95 AD) was a Jewish soldier and historian who lived an exciting and stormy life and inspired an interesting set of mathematical problems. His real name was Joseph ben Matthias. He was born in Jerusalem, studied Hebrew and Greek literature as a child, and then spent three years (between the ages of 16 and 19) as an ascetic with a hermit in the desert. After further study as a member of the Pharisaic sect, he served as a delegate to Emperor Nero, was elected governor of Galilee, and was promoted to the rank of general during the Jewish revolt against Rome in 66. A year later, he was a member of the resistance during the siege of Jodfat, which lasted 47 days. According to Josephus Flavius (1970), doomed soldiers chose to take their own lives rather than be captured by the Romans and suffer an uncertain future. Josephus exclaimed: "Let us entrust our mutual deaths to the decision of lots. Whoever gets the first lot, let the second lot kill him, and so luck will proceed through us all." By chance, fate, or providence," [Josephus] with another was left to endure, to be condemned to the lot, nor, if left to the last to fill his right hand with the blood of his countryman, [Josephus] persuaded him to trust the Roman assurances and he lived as well as he did." Josephus surrendered to Vespasian, who later became the emperor. He travelled extensively with him, and served him and later he served the next emperor, Titus, son of Vespasian, and adopted his family name of Flavius as his own. Josephus was in Jerusalem (as a Roman citizen) during the bloodiest battles of 70 AD, travelled to Rome for the opening of the Colosseum, survived a tragic shipwreck, was married at least three times, and lived a life of excitement and intrigue. More importantly, he wrote several books, including History of the Jewish War, Jewish Antiquities, and Autobiography. His books are valuable and exceptional for witnessing the years of the early Roman empire from the point of view of a person of another culture.

1.2 Formulation of the problem

The problem is named after Flavius Josephus (also Iosephus). According to Josephus' account of the siege of Jodfat, he and the other 40 soldiers were trapped by Roman soldiers in a cave. They preferred serial suicide by drawing lots over capture. Josephus states that by luck, or perhaps the hand of God, he and one other man stayed to the end, surrendering to the Romans rather than killing themselves. This story is told in Book 3, Chapter 8, Part 7 of Joseph's Jewish War (Flavius, 1970). Although the exact wording of Josephus problem varies slightly in different sources (Schumer, 2002), primarily in the starting number of soldiers standing in the circle, from which men will be gradually selected (for example, every third or seventh, etc.) to be killed. The goal is to find the position where the person in question remains the last one. In other words, which position will ensure the person's survival? According to one version of the Josephus problem, 15 Turks and 15 Christians are on board the ship. The ship will surely sink unless half the passengers are thrown overboard. All 30 stand in a circle and decide that every ninth person will be thrown into the sea. The task is to determine where the Christians should line up to ensure that all the Turks are thrown overboard first.

[1] University of South Bohemia in České Budějovice/Faculty of Education, Department of Mathematics, Jeronýmova 10, 371 15 České Budějovice, Czech Republic, troskovec@jcu.cz.
[2] University of South Bohemia in České Budějovice/Faculty of Economics, Department of Applied Mathematics and Informatics, Studentská 13, 370 05 České Budějovice, Czech Republic, chladek@jcu.cz.
[3] University of South Bohemia in České Budějovice/Faculty of Economics, Department of Applied Mathematics and Informatics, Studentská 13, 370 05 České Budějovice, Czech Republic, fiala@jcu.cz.

https://doi.org/10.32725/978-80-7694-053-6.41

In addition to being an interesting historical story, the Josephus problem can be used to solve contemporary and modern tasks. For example, image encryption utilizes the principles of the Josephus problem (Hua, Xu, Jin, & Huang, 2019).

2 Classical formulation and solution

Suppose k people (denoted from 1 to k) stand around the circle. Gradually, every second person is eliminated by their neighbour. In the first step, person number 2 is eliminated. In the second step, number 4, etc. The circle is circled several times until all soldiers are eliminated except for the last survivor. (See figure Nr. 1.) Previously eliminated individuals are no longer counted in the further course of the round. The most classical variant is $k = 41$ (Josephus and 40 other soldiers).

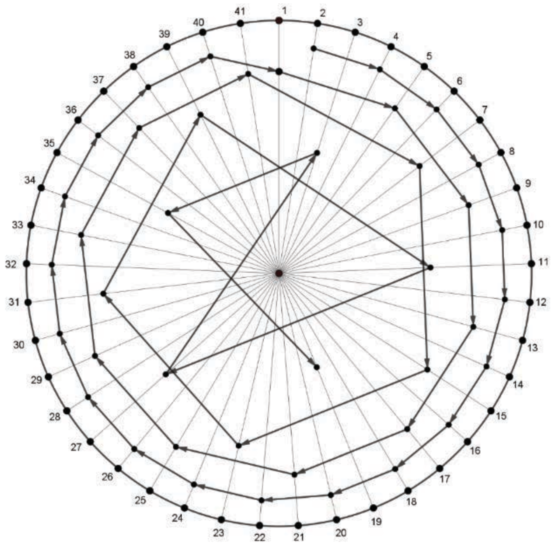

Figure 1 Josephus problem for $k = 41$ and $s = 2$

The solution of the classical formulation is well known; for k soldiers and step $s = 2$, we decompose $k = 2^m + n$ for maximal m so that n is non-negative, and the last soldier standing would be on position $p(k, 2) = 2n + 1$. There are several ways to let students find the solution.

Exercise 1 (Easy, algorithm): Solve the classical Josephus problem with step $s = 2$ and a given number of soldiers k.

By hand, a student can find the solution for a given number of soldiers, but to prove the general formula is somewhat tricky, as shown in the next section. The first attempt would be collecting data. For small k by hand, for higher numbers by an algorithm simulating the process. A detailed algorithm analysis can be found in (Graham, Knuth, & Patashnik, 1989); the authors apply a recurrent algorithm to solve the problem. The following much simpler algorithm is designed with general s and k.

First, initiate variables; all soldiers are alive, and the position of the *Victim* is one step before the first soldier. Then, by *Determining the next victim*, we go forward until we pass s living soldiers. Then, we change the status of the *Victim* to *Not Alive* and write the number of the *Victim*. After repeating k times, the last written number is the survivor.

Iniciate:: Let Number:=k; Step:=s; Victim:=0; Status(i)=Alive for i=1… k.

Determining the next victim: Count:=0. Looked=Victim

Repeat until Count:=s: If Status(Looked+1)=Alive then (Count:=Count+1 and Looked:=Looked+1) else (Looked:=Looked+1). If Looked:=k+1, then Looked:=1.

Victim:=Looked; Status(Victim)=Not Alive; write Victim

Program: Initiate; for i=1 … k Determining the next victim.

Such an algorithm is a good exercise for the first semester of coding and produces the following results.

Table 1 Solution of classical formulation for general k and $s = 2$

Number of soldiers k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Last soldier alive $p(k, 2)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

Exercise 2 (Normal, coding): Code the program to solve the classical Josephus problem with a general step s and a number of soldiers k.

Exercise 3 (Easy, observation): Suggest a formula for general $p(k, 2)$ based on Table 1 for $s = 2$ and k general. Table 1 or further results can be given as data.

Once we obtain the set of results, we can observe sequences of the odd numbers restarted when reaching a power of two, which is precisely the solution described at the beginning of this section. From the engineering point of view, the problem is solved, but mathematicians should demand proof of correctness. The first idea of the proof is that if there are $k = 2^m$ soldiers alive, then clearly the first soldier is a survivor, as in the first round, all the even numbers are eliminated, then there are 2^{m-1} soldiers, number one is starting, and the new position of soldiers can be calculated by the addition of 1 and division by 2 . By this process, we eliminate all numbers, but number one would stay untouched. If there is $k = 2^m + n$ soldiers for $0 \leq n < 2^m$, then we eliminate the first n soldiers, and we can use the previous solution. Which soldier would start after the elimination of n? Clearly, the $2n + 1$-st soldier is the one, as after n steps from the first, we have not finished the first round. This one is in the situation we described before and would be the last one.

Exercise 4 (Normal, theory of numbers): Solve and explain the solution for the classical Josephus problem with step $s = 2$ and number of soldiers $k = 2^m$.

Exercise 5 (Normal, theory of numbers): Based on Exercise 4, solve the classical Josephus problem with step $s = 2$ and number of soldiers $k = 2^m + n$, n non-negative, and $n < 2^m$.

An alternative way to prove the correctness of the solution is by mathematical induction, but the solution is rather tricky; the cases should be separated for k odd and k even, and so-called strong induction should be used. This information can be used to lower the difficulty of following exercise.

Exercise 6 (Hard, mathematical induction): Prove the correctness of the classical Josephus problem with step $s = 2$ and the general number of soldiers k by mathematical induction.

3 Alternative versions

The first natural alternation can be done by alternating the steps ($s = 3$, $s = 4$, etc.). This idea is interesting because it shows the algorithmic and mathematical solutions gap. By algorithm, we can use pseudocode as before to determine the result without any issues, but the result is hard to formulate and prove. It was covered in the step of size 2 (Halbeisen & Hungerbühler, 1997). The general case is described in (Park & Teixeira, 2018).

According to (Schumer, 2002), there exists a general formula. For a given number of soldiers k and given step s, the recurrent formula has the form

$$p(k, s) \equiv (p(k-1, s) + s) \bmod (k).$$

3.1 Fixed step alternative

As these are covered, we may look at the alternative, where the step determines the killed person, but the step will include both living and killed. It means the distance between the killed persons in the original numbering of soldiers is constant. We still denote the number of soldiers k, but the step denoted by s would be the constant distance, which means $s = 2$ would mean every second soldier is killed regardless of living and dead soldiers.

First of all, there is a possibility that some soldiers would be killed multiple times, and some would stay alive. For example, $s = 2$, $k = 2$ means that the second soldier would be killed repeatedly, and the first would remain alive. This leads to the paradox, so let us consider the possible outcomes: The finite number of turns would eliminate all soldiers (as in the case of the non-fixed step), or any number of turns does not eliminate all, and some soldiers survive. Clearly, there is no other option.

Exercise 7 (Normal, theory of numbers): Determine the rule for numbers s and k such that in the fixed step alternative Josephus problem, all or just some soldiers would be killed.

The solution is obvious: after several steps, some of the already killed soldiers would be picked to be killed again, and it starts a cycle. If the cycle is started before all soldiers are killed, then some of them are destined not to be killed. So, the question can be formulated in the language of number theory. If we add s modulo k, do we get 0 earlier than after k steps? This is equivalent to: Is there a common multiple of s and k smaller than sk? This is a trivial question solved by the Euclidean algorithm as if there is any common divider of s and k bigger than 1, then the answer is positive; it means that for s, k not coprime, we eliminate all soldiers, if k and s are coprime than some of the soldier survive the infinite number of steps. Star polygons can be successfully used to model the solution of fixed step alternative Josephus problem. A star polygon is a non-convex polygon formed by connecting each vertex of a given regular k-gon with such non-adjacent vertices whose location from the selected vertex is constant, marked s. For example, we connect the selected vertex with every second (third, fourth, etc.) vertex in the positive or negative sense of rotation. Figure Nr. 2 shows an example of a star pentagon created in a regular pentagon by successively connecting every second vertex, i.e., the sequence 241352. When describing a star polygon, we use the Schläfli symbol $\{k, s\}$ for the number of vertices k and step s.

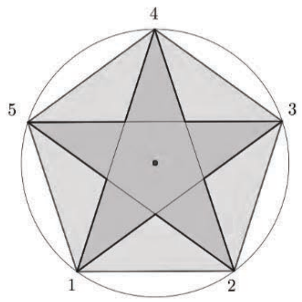

Figure 2 Star polygon of the type $\{5,2\}$

Some star polygons are so-called single lines. If in the Schläfli symbol $\{k, s\}$ of some k-gon there are natural numbers k and s such that k and s are coprime, it will not be possible to sketch the star polygon in one stroke, and it will not be a so-called single line (except for the k-gon with the symbol $\{k, 1\}$, which is always the original non-star k-gon). For example, one cannot create a star decagon $\{10,2\}$ as a single line in a regular decagon because the numbers 10 and 2 are coprime. We will now use the knowledge of single-line star polygons to solve the Josephus problem. For k, s non-coprime, if k soldiers are placed in a circle, and every s -th soldier is killed, the one from the circle standing in the position with the highest mark will remain alive.

Exercise 8 (Easy, polygons, theory of numbers): Find out which star polygons $\{8, s\}$ are single-lines.

We note the possibilities for different steps s. In this version, there are only k possible sizes of s, as $s = k + n$ is similar to the case $s = n$. This is different from the classical case, where the step is similar in the first turn, but after x elimination, the similarity is between $s = k - x + n$ and $s = n$. So the class of similar steps has to have a difference coprime to $k - x$ for all x from 0 to $k - 1$, which leads to the only possible difference being a multiple of $k!$. So, in the classical case, there are $k!$ possible steps, in fixed step alternative, there are only k of them.

3.2 Multiple survivals or multiple victims question

Another question is the position of multiple survivors, both in classical and in the fixed step alternative. The original story includes two survivors. In the medieval tale of the ship with Turks and Christians, one-half of the population survives, and a similar question is given in the story version concerning heritage and children from different marriages (Schumer, 2002). These questions are very rich and can be used as additional exercises for algorithm training or coding and can be expanded further, exceeding the topic of this article. Similarly, a situation where several people are eliminated in one step can be considered. Suppose $k = 2t$, the enemies are near, and there is no time for elimination step by step. Therefore, two eliminations with the same step are realized in parallel. The first elimination starts from soldier number 1; the second starts from soldier number $t + 1$. In the same way, there can be $k = 3t$ soldiers, and triple elimination starts from numbers 1, $t + 1$, and $2t + 1$. The general situation is described in (Yamauchi, Inoue, & Tatsumi, 2009).

3.3 Inverse Josephus problem

As the classical question is for given step s and number of soldiers k, what is the final survivor position $p(k, s)$, we can formulate an inverse question in a way: for the position p of the intended survivor and k the number of soldiers, what should be the step s? Analogously, given a survivor position p and step s, is there a number of soldiers k such that the position p would be the last? As we mentioned, if we have a fixed position and a number of soldiers, there are more values of the step that end in the same position. We have $k!$ classes of possible steps, so we can answer the first question for given k by brute force testing all possibilities with the algorithm. Analogously, but without limitation allowing for covering all cases, we can use the previous algorithm to attempt to find the answer to the second question. Nevertheless, since there is no bound to k, the negative answer would not be complete from the mathematical point of view.

Exercise 9 (Easy, coding): Based on Exercise 2, for given position p and the number of soldiers k, determine if there is a step so p would be the position of the last survivor.

Exercise 10 (Easy, coding): Based on Exercise 2, for given position p and step s, determine if there is a number of soldiers up to 1000, so p would be the position of the last survivor.

Similar questions were asked and partially answered in (Stack Exchange, "Josephus problem"). In general, both questions are still open from the mathematical point of view, and both numerical and analytical solution is of interest.

3.4 Josephus permutations

Let us reformulate the Josephus problem from the point of view of permutations. According to (Schumer, 2002), the Josephus permutation $P(k, s)$ is the permutation created by the Josephus elimination with step s. For example

$$P(5,2) = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,4\,1\,5\,3 \end{pmatrix},$$

since for 5 soldiers and step 2, the first eliminated is number 2, the second eliminated is number 4, the third eliminated is number 1, the fourth eliminated is number 5, and the last is number 3. Naturally, there arises a question of whether all permutations are feasible as Josephus for convenient number s. We have the criteria for the number of all permutations: $k!$ and the number of Josephus permutations: least common multiplier $lcm(1, \dots, k)$. For $k = 3$, $k! = lcm(1, \dots, k)$ holds. It means that for 3 soldiers, every permutation is Josephus. But for $k > 3$ we have $k! > lcm(1, \dots, k)$. Thus, for more than 3 soldiers, some permutations are not Josephus permutations.

Exercise 11 (Easy, permutations)**:** Construct the Josephus permutation for $k = 7$ soldiers and step $s = 3$.

Exercise 12 (Normal, permutations)**:** For 3 soldiers ($k = 3$), consider all 6 permutations. Find appropriate s to construct such Josephus permutations (remember that s has not to be smaller than k).

Note that the number of unique permutations is $k!$, equivalent to the upper bound of possible steps. Nevertheless, as not all permutations are Josephus permutations, we may exclude some classes of possible steps as the merge in order to produce less than $k!$ permutations. This can be used as an efficiency improvement in some previous algorithms, as we do not need to test all $k!$ steps if we can identify the steps generating the same class.

Exercise 13 (Normal, theory of numbers, permutations)**:** For a given number of soldiers, identify steps that generate the same permutation, which means the two different steps determine the same order of victims.

3.5 Stochastic Josephus problem

Exercise 14 (Hard, probability): Number of soldiers k are standing in the circle. Starting with number 1, which is killed, one soldier neighbouring to the last victim randomly (with probability ½ left and ½ right) would be killed until only one soldier would remain. Where to stand in relation to the first victim, so the probability of being the survivor is the highest?

The solution of the previous exercise depends on the precise formulation: If the probability refers to moving one step to the first living soldier in that direction, then after k steps, all soldiers are killed. In this way, we can easily calculate the probability of survival by counting the number of permutations that avoid the position before the last step. For example, the position neighbouring to the starting kill has to be avoided by $k - 1$ repeated steps in the other direction. Positioning one step away allows sequences with one step to this direction and $k - 2$ ones to the opposite direction. The question is transferred to a similar question: What is the probability to toss a coin and get a head $k - 1$ times in the row for the neighbour or to toss $k - 2$ heads and one tail for the position one step away, and so on? This is the well-known situation as for $k - 1 - n$ heads and n tails, the probability is $\binom{k}{k-1-n}2^{-k+1}$. Up to the multiple 2^{-k+1}, we get the row in the Pascal triangle or binomial coefficients $\binom{k}{k-1-n}$. Galton's desk would reveal that the safest position is opposite in the starting one, or $k/2$ positions to any direction.

Exercise 15 (Hard, probability): k soldiers are standing in a circle. Starting with number 1, which is killed, randomly (with probability ½ left and ½ right), one neighbouring position of the soldier is the next victim (it means the soldier on the position is killed, if alive, and the position is regarded as the last victim) until only one soldier would remain. Where to stand in relation to the first victim, so the probability of being the last one is the highest?

The situation is very different, and there is no finite number of turns, as we can repeat the left-right switch for unlimited time. However, we can determine the situation based on symmetries with a surprising outcome: the probability of any position (except for the starting one) to be the last survivor is $(k - 1)^{-1}$. An interesting analogy exists with exercise from the book (Anděl, 2007) about the wolf and sheep. We have 4 positions in the circle. Position 1 (figure Nr. 3 left) represents the wolf (dark dot); the others are sheep (bright dots). The wolf moves to the adjacent space and picks the direction randomly with a probability of ½. If a sheep is in that position, the wolf will eat it; otherwise, he just moves. The movement of the wolf continues according to the same rules. Which of the positions 2, 3, and 4 should the wise sheep occupy so that the wolf comes to it last? Let p_i, $i = 2,3,4$ denote the probability that the wolf is the last to visit position i. We have the equalities $p_2 + p_3 + p_4 = 1$ and according to the symmetry $p_2 = p_4$. Further, let us imagine the situation where, after the first move, the wolf is in position 2 (figure Nr. 3 right). Then, the probability the wolf standing at position 2 visits position 3 the last is the same as the wolf starting from position 1 visits position 2 the last. It follows $p_3 = ½ . p_2 + ½ . p_4$. From previous equalities, we have a surprising solution $p_2 = p_3 = p_4 = \frac{1}{3}$.

Other variants can be added, either by randomization of the step or by randomisation of the direction in different ways. Such a question would lead us out of the topic and deserve deep and separate analysis.

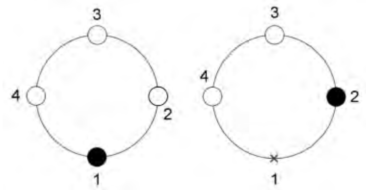

Figure 3 Exercise about the wolf and sheep for $k = 4$. Left: default situation, right: the first step of the wolf in the circle.

4 Conclusions

We intend to reveal how a simple historical mention can lead to developing a series of questions and inspire examinations and application of several methods. Note that our study does not exhaust the topic and techniques, but we would need both to use more advanced methods and does not lead to the goal of a lecturer/student-friendly paper. Several other exciting inspirations arise from a non-mathematical situation; examples include the Monty-Hall problem from a TV show, airplane plating optimisation by Abraham Wald, or permutation algebra needed to break the Enigma ciphering.

References

Anděl, J. (2007). *Matematika náhody.* Praha: Matfyzpress.

Graham, R. L., Knuth, D. E., & Patashnik, O. (1989). *Concrete Mathematics: A Foundation for Computer Science.* Addison Wesley, 8-20.

Halbeisen, L., & Hungerbühler, N. (1997). The Josephus problem, *Journal de théorie des nombres de Bordeaux,* 9(2), 303-318. http://www.numdam.org/item/JTNB_1997__9_2_303_0/, https://doi:10.5802/jtnb.204

Hua, Z., Xu, B., Jin, F., & Huang, H. (2019). Image Encryption Using Josephus Problem and Filtering Diffusion, in *IEEE Access*, vol. 7, pp. 8660-8674, 2019, DOI 10.1109/ACCESS.2018.2890116.

Flavius, J. (1970). *The Great Roman-Jewish War: A.D.,* 66-70, Peter Smith, Gloucester, 138-139.

Park, J.-W., & Teixeira, R. (2018). Serial execution Josephus Problem. The *Korean Journal of Mathematics*, 26(1), 1-7. DOI 10.11568/kjm.2018.26.1.1.

Schumer, P. (2002). The Josephus Problem: Once More Around. *Mathematics Magazine,* 75(1), 12-17. DOI 10.2307/3219179.

Yamauchi, T., Inoue, T., & Tatsumi, S. (2009). Josephus Problem Under Various Moduli. *Rose-Hulman Undergraduate Mathematics Journal*, 10(1), 10. https://scholar.rose-hulman.edu/rhumj/vol10/iss1/10.

Stack Exchange, Mathematics [website], "*Inverse Josephus problem".*
https://math.stackexchange.com/questions/143380/inverse-josephus-problem